ACCESS TO SCIENCE, ENGINEERING AND AGRICULTURE: MATHEMATICS 1

MATH00030

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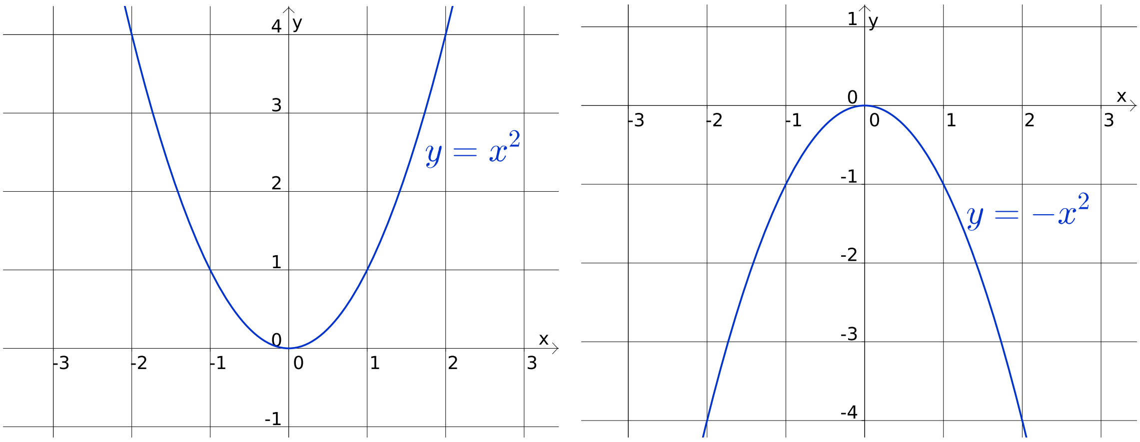
Contents

3. Quadratic Equations

3.1. Graphs of Quadratic Functions.

In Chapter 2 we looked at straight lines, the simplest type of graph. In this chapter we will move on to quadratic functions, the graphs of which which may be regarded as the next simplest type. Recall that the general equation of a straight line (apart from vertical lines) is $y = mx + c$, where m is the slope and c is the y-intercept. One way of looking at this is that the line needs two numbers m and c (we sometimes call these numbers *parameters*) to completely specify it. The general equation of a quadratic curve is $y = ax^2 + bx + c$, so we see that we need three parameters, a, b and c to completely specify it. Note that we have to have $a \neq 0$ for otherwise the equation is linear.

We will have to wait until Section 3.4 to be able to go from an equation to sketching a graph but we will start off in this section by plotting some graphs and looking at some of the main features. The effects that a and c have on the graph are much easier to explain then the effect that b has on the graph, so we will start out by considering equations where $b = 0$. First, let us examine Figure 1.



(A) The graph of $y = x^2$.

(B) The graph of $y = -x^2$.

FIGURE 1. The effect of changing a from positive to negative.

As you can see the graph of $y = x^2$ (where $a = 1$, $b = 0$ and $c = 0$) is U-shaped and passes through the origin. If we change $a = 1$ to $a = -1$ but keep b and c zero, then we get the graph of $y = -x^2$ which is an upside down U-shape, still passing through the origin.

Remark 3.1.1. No matter what the values of b and c are, if a is positive then the graph will always be U-shaped and if a is negative then the graph will always be shaped like an upside down U. The width of the U will vary with a though and we will examine this in Figures 3 and 4 below.

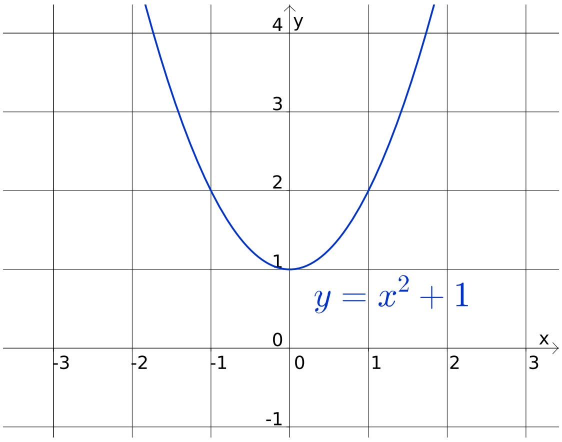
Remark 3.1.2. Note that all the graphs in this section have the same scale to aid comparison.

Next let us fix $a = 1$ and $b = 0$ and see what effect changing c has on the graph. This is shown in Figure 2.

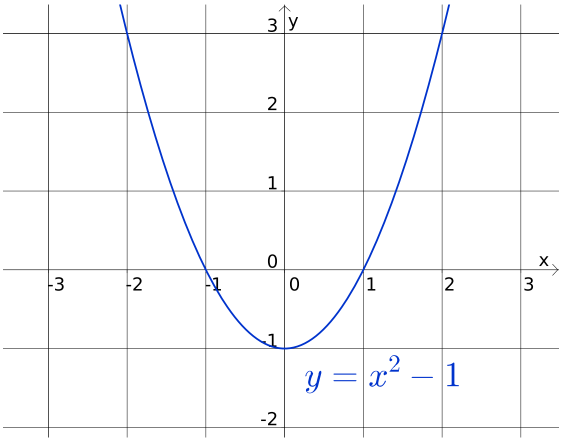
In both cases the shape of the graph remains the same as in Figure 1A but when $c = 1$ (Figure 2A) the graph has been shifted up by one unit and when $c = -1$ (Figure 2B) the graph has been shifted down by one unit.

Remark 3.1.3. In particular note that in both cases c gives the y-intercept. In fact this remains true no matter what the values of a and b are. We can see this by noting that when $x = 0$, $ax^2 + bx + c = a(0) + b(0) + c = c$. That is, when $x = 0$, $y = c$, so c is the y-intercept.

It is also the case that no matter what the values of a and b are, then increasing c by a certain number moves the graph upwards by that number of units and decreasing c by a certain number moves the graph downwards by that number of units. We



(A) The graph of $y = x^2 + 1$.

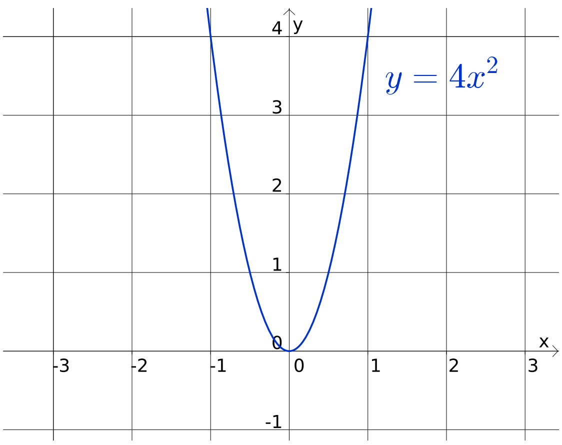


(B) The graph of $y = x^2 - 1$.

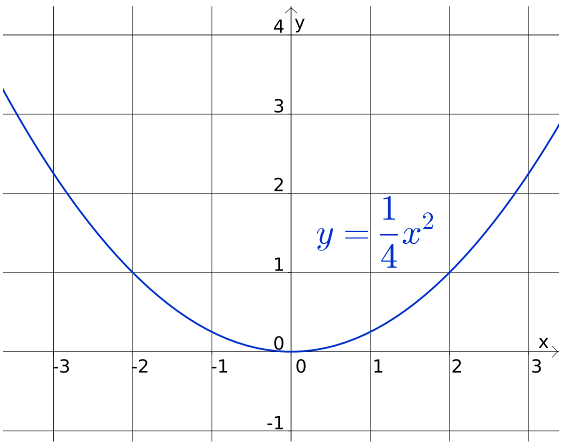
FIGURE 2. The effect of changing c.

can see this, since if a and b are held fixed, then changing c will change y by exactly the same amount.

Next let us see what happens when we have different values of positive a; this is shown in Figure 3.



(A) The graph of $y = 4x^2$.



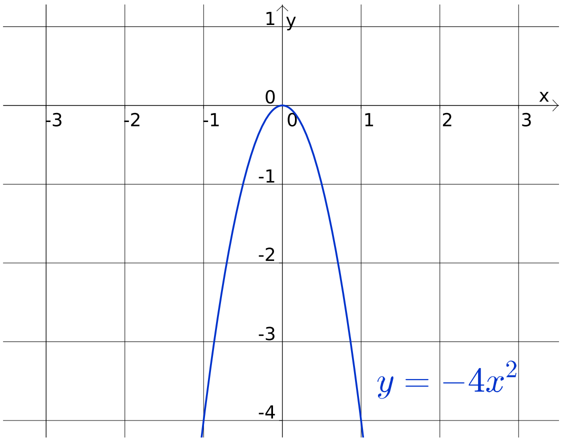
(B) The graph of $y = \frac{1}{4}x^2$.

FIGURE 3. The effect of changing a when a is positive.

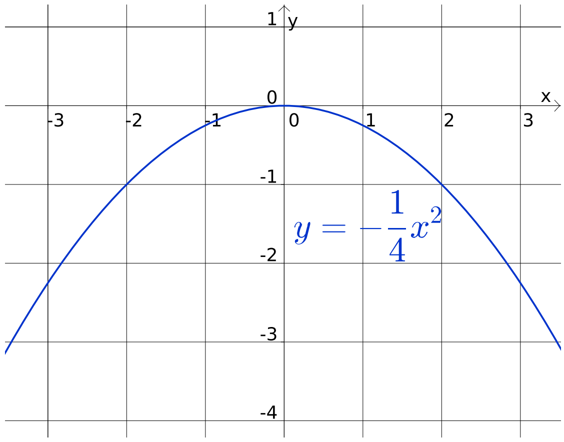
We see in Figure 3A that if a is increased then the graph becomes steeper and we see in Figure 3A that if a is decreased then the graph becomes shallower.

Remark 3.1.4. Note that this remains the case no matter what the values of b and c are.

Finally let us do the same thing with a negative; this is shown in Figure 4.



(A) The graph of $y = -4x^2$.



(B) The graph of $y = -\dfrac{1}{4}x^2$.

FIGURE 4. The effect of changing a when a is negative.

Here we see similar behaviour as when a was positive. When the magnitude of a is increased then the graph becomes steeper and when the magnitude of a is decreased then the graph becomes shallower.

I have prepared a GeoGebra worksheet which will enable you to change the values of a, b and c using 'sliders' and see the result on the graph in real time. It can be found at http://www.ucd.ie/msc/access/graphofaquadraticfunction/. I would recommend that you have a play around with this worksheet since it makes it much easier to see what happens when you can see the graph changing as you move the slider. Note that as usual, you can always reset the graph to its starting position by clicking on the icon in the top right hand corner of the worksheet.

3.2. **Completing the Square and the Quadratic Formula.**

As was the case with straight lines, we want to be able to sketch quadratic graphs and solve quadratic equations. There are different methods we can use to do this. For example, we can use the techniques of calculus to help us sketch quadratic graphs (and indeed many other sorts of graphs as well). However we will not study this method until the second trimester, and for the moment we will use the technique known as *completing the square*. This has the advantage of enabling us to not only sketch quadratic graphs but also to solve quadratic equations.

3.2.1. *Completing the Square*.

The idea behind the technique is to simplify the equation $y = ax^2 + bx + c$ by re-writing it as $y = dz^2 + e$ where d and e are numbers and z is a new variable. In general z will be x plus or minus a number. The advantage in writing the equation in this form is that it helps us to sketch the graph of $y = ax^2 + bx + c$ and also solve the equation $y = ax^2 + bx + c = 0$.

Before we look at sketching graphs and solving equations, let us do a few examples of writing quadratic expressions in what is known as *completed square form.*

Example 3.2.1. Write $x^2 + 2x + 2$ in completed square form.
Here we have

$$x^2 + 2x + 2 = [x^2 + 2x] + 2 = [(x + 1)^2 - 1] + 2 = (x + 1)^2 + 1.$$

Note what we have done, we have taken the $x^2 + 2x$ part and have written it as a square of $x + 1$ minus a constant. This constant is then subtracted from the 2 we started with. Also note that the 1 in $x+1$ comes from half the coefficient of x in the original expression, i.e., 2. Finally the constant -1 arises since $(x+1)^2 = x^2+2x+1$, so that $x^2 + 2x = (x + 1)^2 - 1$. Another way of looking at this is that in writing $(x + 1)^2$, we have added 1 onto $x^2 + 2x$, so to balance everything out, we have to subtract it off again.

Also note that we have written $x^2 + 2x + 2$ as $dz^2 + e$, where $d = 1$, $e = 1$ and $z = x + 1$.

Example 3.2.2. Write $x^2 - 3x + 5$ in completed square form.
In this case

$$x^2 - 3x + 5 = [x^2 - 3x] + 5 = \left[\left(x - \frac{3}{2}\right)^2 - \frac{9}{4}\right] + 5 = \left(x - \frac{3}{2}\right)^2 + \frac{11}{4}.$$

Note here that the general procedure is the same as in Example 3.2.1. We first consider just $x^2 - 3x$ and then write down $\left(x - \frac{3}{2}\right)^2$ where $-\frac{3}{2}$ is half of the coefficient of x in $x^2 - 3x$. However this means we have added in $\left(-\frac{3}{2}\right)^2 = \frac{9}{4}$, so we have to subtract it off again.

Warning 3.2.3. Note that $\left(-\frac{3}{2}\right)^2$ is positive, so we have to subtract $\frac{9}{4}$, not add it.

Things become slightly more complicated when the coefficient of x^2 is not 1.

Example 3.2.4. Write $2x^2 - 5x - 3$ in completed square form.
In this case

$$\begin{aligned} 2x^2 - 5x - 3 &= 2\left\{x^2 - \frac{5}{2}x - \frac{3}{2}\right\} \\ &= 2\left\{\left[x^2 - \frac{5}{2}x\right] - \frac{3}{2}\right\} \\ &= 2\left\{\left[\left(x - \frac{5}{4}\right)^2 - \frac{25}{16}\right] - \frac{3}{2}\right\} \\ &= 2\left\{\left(x - \frac{5}{4}\right)^2 - \frac{49}{16}\right\} \\ &= 2\left(x - \frac{5}{4}\right)^2 - \frac{49}{8}. \end{aligned}$$

The main thing to note here is that we take the factor of 2 out at the start and then proceed with what we have inside the curly brackets as we did in Examples 3.2.1 and 3.2.2. Note we could also leave the -3 alone and add it at the end but this doesn't make much difference to the difficulty of the calculation one way or the other.

Example 3.2.5. Write $-\frac{1}{2}x^2 + \frac{1}{3}x + \frac{2}{5}$ in completed square form.
In this case

$$\begin{aligned} -\frac{1}{2}x^2 + \frac{1}{3}x + \frac{2}{5} &= -\frac{1}{2}\left\{x^2 - \frac{2}{3}x - \frac{4}{5}\right\} \\ &= -\frac{1}{2}\left\{\left[x^2 - \frac{2}{3}x\right] - \frac{4}{5}\right\} \\ &= -\frac{1}{2}\left\{\left[\left(x - \frac{1}{3}\right)^2 - \frac{1}{9}\right] - \frac{4}{5}\right\} \\ &= -\frac{1}{2}\left\{\left(x - \frac{1}{3}\right)^2 - \frac{41}{45}\right\} \\ &= -\frac{1}{2}\left(x - \frac{1}{3}\right)^2 + \frac{41}{90}. \end{aligned}$$

Remark 3.2.6. As always in mathematics, when we have obtained our answer, it is good practice to try and check it in some way. For these problems there is an easy check; we just have to multiply out the completed square form and make sure we get the original expression.

As I remarked at the start of this section, completing the square helps us to sketch quadratic graphs and also solve quadratic equations. We will look at sketching graphs in Section 3.4 but here we will concentrate on solving equations. In general we will want to solve equations of the form $ax^2 + bx + c = 0$. Before tackling the general case, we will do some examples.

Example 3.2.7. Solve the equation $x^2 - 4x + 3 = 0$.
We will first complete the square on $x^2 - 4x + 3$.

$$x^2 - 4x + 3 = [x^2 - 4x] + 3 = [(x-2)^2 - 4] + 3 = (x-2)^2 - 1.$$

Since $x^2 - 4x + 3 = (x-2)^2 - 1$, we can rewrite the equation $x^2 - 4x + 3 = 0$ as $(x-2)^2 - 1 = 0$. We have now done all the hard work since this last equation is easy to solve

$$(x-2)^2 - 1 = 0 \Rightarrow (x-2)^2 = 1 \Rightarrow x - 2 = \pm 1.$$

Now if $x - 2 = -1$ then $x = 1$ and if $x - 2 = 1$ then $x = 3$.
Thus the solutions of the equation are $x = 1$ and $x = 3$.

Remark 3.2.8. We should now go back and check that $x = 1$ and $x = 3$ are indeed solutions of $x^2 - 4x + 3 = 0$:

$$1^2 - 4(1) + 3 = 1 - 4 + 3 = 0 \quad \text{and} \quad 3^2 - 4(3) + 3 = 9 - 12 + 3 = 0.$$

Thus $x = 1$ and $x = 3$ are solutions of $x^2 - 4x + 3 = 0$.

Example 3.2.9. Solve the equation $x^2 + 6x + 5 = 0$.
We will first complete the square on $x^2 + 6x + 5$.

$$x^2 + 6x + 5 = [x^2 + 6x] + 5 = [(x+3)^2 - 9] + 5 = (x+3)^2 - 4.$$

We can now rewrite the equation $x^2 + 6x + 5 = 0$ as $(x+3)^2 - 4 = 0$.
This can then be solved as follows:

$$(x+3)^2 - 4 = 0 \Rightarrow (x+3)^2 = 4 \Rightarrow x + 3 = \pm 2.$$

Now if $x + 3 = -2$ then $x = -5$ and if $x + 3 = 2$ then $x = -1$.
Thus the solutions of the equation are $x = -5$ and $x = -1$.

Example 3.2.10. Solve the equation $x^2 + 6x + 9 = 0$.
We will first complete the square on $x^2 + 6x + 9$.

$$x^2 + 6x + 9 = [x^2 + 6x] + 9 = [(x+3)^2 - 9] + 9 = (x+3)^2.$$

We can now rewrite the equation $x^2 + 6x + 9 = 0$ as $(x+3)^2 = 0$.
This can then be solved as follows:

$$(x+3)^2 = 0 \Rightarrow x + 3 = 0 \Rightarrow x = -3.$$

Thus the solution is $x = -3$.
Note that in contrast to the first two examples, this equation only has one solution.

So far all the solutions have been integers. Now let us do an example where the solutions will involve square roots.

Example 3.2.11. Solve the equation $2x^2 - 4x + 1 = 0$.
We will first complete the square on $2x^2 - 4x + 1$.

$$\begin{aligned}
2x^2 - 4x + 1 &= 2\left\{x^2 - 2x + \frac{1}{2}\right\} \\
&= 2\left\{[x^2 - 2x] + \frac{1}{2}\right\} \\
&= 2\left\{[(x-1)^2 - 1] + \frac{1}{2}\right\} \\
&= 2\left\{(x-1)^2 - \frac{1}{2}\right\} \\
&= 2(x-1)^2 - 1.
\end{aligned}$$

We can now rewrite the equation $2x^2 - 4x + 1 = 0$ as $2(x-1)^2 - 1 = 0$.
This can then be solved as follows:

$$2(x-1)^2 - 1 = 0 \Rightarrow 2(x-1)^2 = 1 \Rightarrow (x-1)^2 = \frac{1}{2} \Rightarrow x - 1 = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2}.$$

Now if $x - 1 = -\frac{\sqrt{2}}{2}$ then $x = 1 - \frac{\sqrt{2}}{2}$ and if $x - 1 = \frac{\sqrt{2}}{2}$ then $x = 1 + \frac{\sqrt{2}}{2}$.
Thus the solutions of the equation are $x = 1 - \frac{\sqrt{2}}{2}$ and $x = 1 + \frac{\sqrt{2}}{2}$.

3.2.2. *The Quadratic Formula*.

For those of you who have met the quadratic formula before, the form of the above solutions may ring a bell. In fact we have all the tools we need to derive the quadratic formula and we will do now.

Theorem 3.2.12 (The Quadratic Formula)**.** *If a, b and c are constants with $a \neq 0$, then the equation $ax^2 + bx + c = 0$ has solutions $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.*

Proof. We will first complete the square on $ax^2 + bx + c$.

$$
\begin{aligned}
ax^2 + bx + c &= a\left\{x^2 + \frac{b}{a}x + \frac{c}{a}\right\} \quad \text{(here we need } a \neq 0\text{)}\\
&= a\left\{\left[x^2 + \frac{b}{a}x\right] + \frac{c}{a}\right\}\\
&= a\left\{\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right] + \frac{c}{a}\right\}\\
&= a\left\{\left(x + \frac{b}{2a}\right)^2 + \frac{-b^2 + 4ac}{4a^2}\right\}\\
&= a\left\{\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right\}\\
&= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}. \qquad (1)
\end{aligned}
$$

We can now rewrite the equation $ax^2 + bx + c = 0$ as $a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a} = 0$.

This can then be solved as follows:

$$
\begin{aligned}
a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a} = 0 &\Rightarrow a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a}\\
&\Rightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}\\
&\Rightarrow x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}\\
&\Rightarrow x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}\\
&\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}
$$

□

Theorem 3.2.12 enables us to solve any quadratic equation. Here are a couple of examples.

Example 3.2.13. Solve the equation $3x^2 - 3x - 2 = 0$.
In this case $a = 3$, $b = -3$ and $c = -2$. Hence the solutions of the equations are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(3)(-2)}}{2(3)} = \frac{3 \pm \sqrt{9 + 24}}{6} = \frac{3 \pm \sqrt{33}}{6}.$$

Warning 3.2.14. Remember there is a minus sign before the b, so if b is negative, $-b$ will be positive. Also note that b^2 can never be negative.

Example 3.2.15. Solve the equation $-4x^2 + 2x + 3 = 0$.
In this case $a = -4$, $b = 2$ and $c = 3$. Hence the solutions of the equations are

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-2 \pm \sqrt{2^2 - 4(-4)(3)}}{2(-4)} \\ &= \frac{-2 \pm \sqrt{4 + 48}}{-8} \\ &= \frac{2 \pm \sqrt{52}}{8} \\ &= \frac{1 \pm \sqrt{13}}{4}. \end{aligned}$$

3.3. **Real and Complex Roots** .

Some of you who have met quadratic equations before may be thinking 'hang on a minute, all the equations so far have real solutions, isn't he cheating?'. If you thought this then you would be correct. I have specially chosen all the equations so far to have real numbers as their solutions. However this will not always be the case. The problem occurs when $b^2 - 4ac$ in the quadratic formula is negative. If we square any real number, we get a non-negative real number, so put another way, a real number can't be the square root of a negative number. So if $b^2 - 4ac < 0$, there can be no real solutions of the equation $ax^2 + bx + c = 0$.

The way around this problem is to introduce *complex numbers*. We will devote a whole chapter to complex numbers in the second trimester but for the moment we will just define what a complex number is and show that if we don't have any real solutions of a quadratic equation then we will always have two complex ones.

Definition 3.3.1 (Complex number)**.** A *complex number* is a number of the form $a + bi$ where a and b are real numbers and i has the property that $i^2 = -1$.

If $b^2 - 4ac < 0$, then in order to put the solutions we obtain from Theorem 3.2.12 in the form $a + bi$, we only have to know how to deal with the square roots of negative numbers. Luckily Theorem 1.2.18 from Chapter 1 still holds in this case. So if x is a positive number (so $-x$ is negative) then $\sqrt{-x} = \sqrt{x(-1)} = \sqrt{x} \cdot \sqrt{-1} = \sqrt{x} \cdot i$.

Here are a couple of examples.

Example 3.3.2. Solve the equation $x^2 + 2x + 2 = 0$.
In this case $a = 1$, $b = 2$ and $c = 2$. Hence the solutions of the equations are

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2(1)} \\ &= \frac{-2 \pm \sqrt{4 - 8}}{2} \\ &= \frac{-2 \pm \sqrt{-4}}{2} \\ &= \frac{-2 \pm \sqrt{4}i}{2} \\ &= \frac{-2 \pm 2i}{2} \\ &= -1 \pm i. \end{aligned}$$

Example 3.3.3. Solve the equation $-3x^2 + 3x - 4 = 0$.
In this case $a = -3$, $b = 3$ and $c = -4$. Hence the solutions of the equations are

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-3 \pm \sqrt{3^2 - 4(-3)(-4)}}{2(-3)} \\ &= \frac{-3 \pm \sqrt{9 - 48}}{-6} \\ &= \frac{-3 \pm \sqrt{-39}}{-6} \\ &= \frac{-3 \pm \sqrt{39}i}{-6} \\ &= \frac{1}{2} \pm \frac{\sqrt{39}}{6}i. \end{aligned}$$

Remark 3.3.4. Before we finish this section, we will note that if a quadratic equation $ax^2 + bx + c = 0$ has no real solutions then $b^2 - 4ac < 0$, so that $4ac - b^2 > 0$.

Hence the solutions are

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-b \pm \sqrt{(4ac - b^2)(-1)}}{2a} \\ &= \frac{-b \pm \sqrt{4ac - b^2} i}{2a} \\ &= -\frac{b}{2a} \pm \frac{\sqrt{4ac - b^2}}{2a} i. \end{aligned}$$

Since $4ac - b^2 > 0$ this means that in this case there are two complex solutions. Thus for a quadratic equation $ax^2 + bx + c = 0$ there are three different possibilities:

- If $b^2 - 4ac > 0$ then there are two real solutions.
- If $b^2 - 4ac = 0$ then there is one real solution.
- If $b^2 - 4ac < 0$ then there are two complex solutions. These are said to be in *complex conjugate* pairs but don't worry about this too much at the moment; we will return to study complex numbers in more detail in the second trimester.

The number $b^2 - 4ac$ is called the *discriminant* of the equation, since it discriminates between the different possibilities.

3.4. **Sketching Quadratic Graphs**.

In this section we will use the expertise we have gained in solving quadratic equations to help us sketch quadratic graphs. However before we can proceed we need another technique. When sketching quadratic graphs we need the following information:

(1) We need to know where the graph cuts the x-axis.
(2) We need to know where the graph cuts the y-axis.
(3) We need to know is it U-shaped or is it shaped like an upside down U.
(4) We need to know where the graph has its lowest point (if it is U-shaped) or its highest point (if it is shaped like an upside down U).

Solving quadratic equations helps us with (1), since the points where the graph of $y = ax^2 + bx + c$ cuts the x-axis are the points where $y = 0$, that is where $ax^2 + bx + c = 0$. Note this means that if there are two real solutions of $ax^2 + bx + c = 0$, then the graph cuts the x-axis in two places, if there is one real solution then the graph cuts the x-axis in one place (it just touches the x-axis) and if there are no real solutions, then the graph does not touch the x-axis.

Next, (2) is easy, since to find the y-coordinate of the point where the graph cuts the y-axis, we simply substitute $x = 0$ into $y = ax^2 + bx + c$ and obtain $y = c$. Item (3) is also easy; if $a > 0$ the the graph is U-shaped and if $a < 0$ then it is shaped like an upside down U.

We still have to deal with (4) and it is this that we will look at now. The key is the completed square form. In Equation (1) we showed that

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}. \tag{2}$$

Now $\left(x + \frac{b}{2a}\right)^2$ is always non-negative and equals zero when $x + \frac{b}{2a} = 0$, that is when $x = -\frac{b}{2a}$. If we examine Equation (2), this means that if $a > 0$ then $y = ax^2 + bx + c$ is at is minimum when $x = -\frac{b}{2a}$ with this minimum being $-\frac{b^2 - 4ac}{4a}$. Similarly if $a < 0$ then $y = ax^2 + bx + c$ is at is maximum when $x = -\frac{b}{2a}$ with this maximum being $-\frac{b^2 - 4ac}{4a}$. So in either case the *turning point* of the graph lies at $\left(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a}\right)$.

Remark 3.4.1. There is another way of finding the turning point using the techniques of calculus but while we will start to study calculus in the first trimester, we won't look at finding turning points using calculus until the second trimester. This will be a very valuable technique though, since it can be used to find the turning points of a much larger class of graphs than just quadratic graphs.

We now have all the tools we need to sketch some graphs of quadratic function, so let us do some examples.

Example 3.4.2. Sketch the graph of the function $y = x^2 - x - 2$.
We will first find where the graph cuts the x-axis by solving the equation $y = x^2 - x - 2 = 0$. Using the quadratic formula with $a = 1$, $b = -1$ and $c = -2$, we obtain

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-2)}}{2(1)} \\ &= \frac{1 \pm \sqrt{1 + 8}}{2} \\ &= \frac{1 \pm \sqrt{9}}{2} \\ &= \frac{1 \pm 3}{2}. \end{aligned}$$

Thus the graph cuts the x-axis when $x = -1$ and when $x = 2$.
Next, when $x = 0$, $y = -2$, so the graph cuts the y-axis when $y = -2$.
We also know the graph is U-shaped since $a > 0$.

Finally, the turning point is given by

$$\left(-\frac{b}{2a}, -\frac{b^2-4ac}{4a}\right) = \left(-\frac{-1}{2(1)}, -\frac{(-1)^2-4(1)(-2)}{4(1)}\right) = \left(\frac{1}{2}, -\frac{9}{4}\right).$$

We now have enough information to sketch the graph and I have shown this in Figure 5. Note that I have not included the grid lines or scale on the axes. This is because a sketch is a summary of the main features and shape of the graph (the x and y intercepts, the turning point and whether it is shaped like a U or an upside down U), it is not a plot of the graph. If I was doing the sketch by hand I would not have to worry about calculating exact values of any other points apart from the ones I have already calculated.

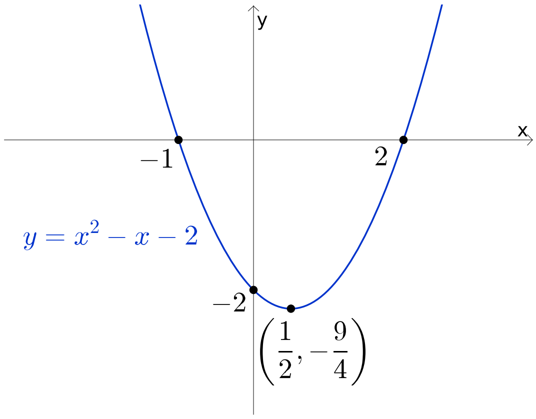


FIGURE 5. A sketch of the graph of the function $y = x^2 - x - 2$.

Example 3.4.3. Sketch the graph of the function $y = -2x^2 + 3x + 1$.
We will first find where the graph cuts the x-axis by solving the equation $y = -2x^2 + 3x + 1 = 0$. Using the quadratic formula with $a = -2$, $b = 3$ and $c = 1$, we obtain

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\ &= \frac{-3 \pm \sqrt{3^2-4(-2)(1)}}{2(-2)} \\ &= \frac{-3 \pm \sqrt{9+8}}{-4} \\ &= \frac{-3 \pm \sqrt{17}}{-4} \\ &= \frac{3}{4} \pm \frac{\sqrt{17}}{4}. \end{aligned}$$

Thus the graph cuts the x-axis when $x = \frac{3}{4} + \frac{\sqrt{17}}{4}$ and when $x = \frac{3}{4} - \frac{\sqrt{17}}{4}$.
Next, when $x = 0$, $y = 1$, so the graph cuts the y-axis when $y = 1$.
We also know the graph is shaped like an upside down U since $a < 0$.
Finally, the turning point is given by

$$\left(-\frac{b}{2a}, -\frac{b^2-4ac}{4a}\right) = \left(-\frac{3}{2(-2)}, -\frac{3^2-4(-2)(1)}{4(-2)}\right) = \left(\frac{3}{4}, \frac{17}{8}\right).$$

We now have all the information we need and I have sketched the graph in Figure 6A below.

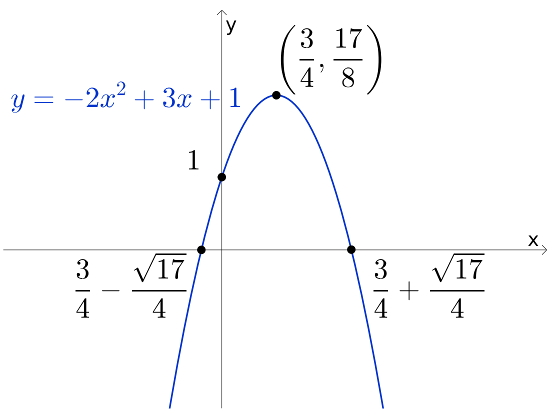
Example 3.4.4. Sketch the graph of the function $y = 3x^2 + 3x + 1$.
We will first find where the graph cuts the x-axis by solving the equation $y = 3x^2 + 3x + 1 = 0$. Using the quadratic formula with $a = 3$, $b = 3$ and $c = 1$, we obtain

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-3 \pm \sqrt{3^2 - 4(3)(1)}}{2(3)} \\ &= \frac{-3 \pm \sqrt{9 - 12}}{6} \\ &= \frac{-3 \pm \sqrt{-3}}{6}. \end{aligned}$$

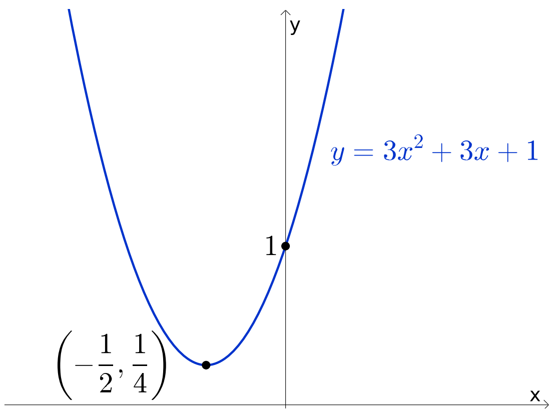
At this point we can stop since the $\sqrt{-3}$ means that the equation $3x^2 + 3x + 1 = 0$ does not have any real solutions, it only has complex ones. This means that the graph does not cut the x-axis.
Next, when $x = 0$, $y = 1$, so the graph cuts the y-axis when $y = 1$.
We also know the graph is U-shaped since $a > 0$.
Finally, the turning point is given by

$$\left(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a}\right) = \left(-\frac{3}{2(3)}, -\frac{3^2 - 4(3)(1)}{4(3)}\right) = \left(-\frac{1}{2}, \frac{1}{4}\right).$$

We now have all the information we need and I have sketched the graph in Figure 6B.



(A) A sketch of the graph of the function $y = -2x^2 + 3x + 1$.



(B) A sketch of the graph of the function $y = 3x^2 + 3x + 1$.

FIGURE 6

3.5. Factorizing Quadratic Expressions.

In the last section of this chapter we will look at factorizing quadratic expressions. There are two ways to do this; firstly we can use the quadratic formula and secondly we can do it ‘by inspection’.

Let us first do using the quadratic formula. In general what we want to do is to express $ax^2 + bx + c$ as $(dx + e)(fx + g)$. However it will be enough to express $ax^2 + bx + c$ as $a(x + e)(x + g)$ since we can then split the a among the factors $x + e$ and $x + g$ as we want. However in order to use the quadratic formula, we have to relate the problem of expressing $ax^2 + bx + c$ as $a(x + e)(x + g)$ to the problem of solving the equation $ax^2 + bx + c = 0$.

Let us suppose that we have $ax^2 + bx + c = a(x + e)(x + g)$. Then $ax^2 + bx + c = 0$ is equivalent to $a(x + e)(x + g) = 0$ (i.e. the solutions are the same). However if $a(x + e)(x + g) = 0$ then, since $a \neq 0$, we must have either $x + e = 0$ or $x + g = 0$, so that the solutions are $x = -e$ and $x = -g$. So given the solutions of $ax^2 + bx + c = 0$ we also know the values of $-e$ and $-g$. As always, some examples will make things clearer.

Example 3.5.1. Factorize $x^2 - 2x - 8$.
We will first use the quadratic formula to solve the equation $x^2 - 2x - 8 = 0$. Since $a = 1$, $b = -2$ and $c = -8$, we obtain

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-8)}}{2(1)} \\ &= \frac{2 \pm \sqrt{4 + 32}}{2} \\ &= \frac{2 \pm \sqrt{36}}{2} \\ &= \frac{2 \pm 6}{2} \\ &= 1 \pm 3. \end{aligned}$$

Thus $x = -2$ and $x = 4$ are the solutions.
Since $a = 1$ we can now write $x^2 - 2x - 8 = (x + 2)(x - 4)$.
Of course, at this stage it is a good idea to multiply out $(x + 2)(x - 4)$ and check we do in fact get $x^2 - 2x - 8$.

Warning 3.5.2. Always remember that we have to change the signs of the solutions of the equation $ax^2 + bx + c = 0$ to get the numbers we put into the factorization.

Example 3.5.3. Factorize $x^2 - 6x + 9$.
We will first use the quadratic formula to solve the equation $x^2 - 6x + 9 = 0$. Since

$a = 1$, $b = -6$ and $c = 9$, we obtain

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(9)}}{2(1)} \\ &= \frac{6 \pm \sqrt{36 - 36}}{2} \\ &= \frac{6 \pm \sqrt{0}}{2} \\ &= 3. \end{aligned}$$

Thus $x = 3$ is the only solution. In this case we say $x = 3$ is a *repeated* root and so $x^2 - 6x + 9 = (x - 3)(x - 3) = (x - 3)^2$.

Example 3.5.4. Factorize $3x^2 - 5x + 1$.
Again we will first use the quadratic formula to solve the equation $3x^2 - 5x + 1 = 0$. Since $a = 3$, $b = -5$ and $c = 1$, we obtain

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(1)}}{2(3)} \\ &= \frac{5 \pm \sqrt{25 - 12}}{6} \\ &= \frac{5 \pm \sqrt{13}}{6}. \end{aligned}$$

Thus $x = \dfrac{5 - \sqrt{13}}{6}$ and $x = \dfrac{5 + \sqrt{13}}{6}$ are the solutions.
Since $a = 3$ we can now write $3x^2 - 5x + 1 = 3\left(x - \dfrac{5 - \sqrt{13}}{6}\right)\left(x - \dfrac{5 + \sqrt{13}}{6}\right)$.
Of course, if we wanted to, we could bring the 3 inside one of the brackets. For example $3x^2 - 5x + 1 = \left(3x - \dfrac{5 - \sqrt{13}}{2}\right)\left(x - \dfrac{5 + \sqrt{13}}{6}\right)$.

Example 3.5.5. Factorize $x^2 + x + 8$.
We will first use the quadratic formula to solve the equation $x^2 + x + 8 = 0$. Since

$a = 1$, $b = 1$ and $c = 8$, we obtain

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-1 \pm \sqrt{1^2 - 4(1)(8)}}{2(1)} \\ &= \frac{-1 \pm \sqrt{1 - 32}}{2} \\ &= \frac{-1 \pm \sqrt{-31}}{2} \\ &= \frac{-1 \pm \sqrt{31}i}{2}. \end{aligned}$$

Thus the solutions are complex in this case. Depending on the sort of mathematics we are studying, we may or may not regard this as a factorization. In this course we will say that $x^2 + x + 8$ cannot be factorized.

Remark 3.5.6. If you are asked to sketch the graph of $y = ax^2 + bx + c$ or factorize $ax^2 + bx + c$, then it can be a good idea to first calculate the discriminant $b^2 - 4ac$. If it turns out to be negative then we know that the graph of $y = ax^2 + bx + c$ does not intersect the x-axis or that $ax^2 + bx + c$ cannot be factorized.

Finally we will look at factorization 'by inspection'. This just means that we look at $ax^2 + bx + c$ and decide that it factorizes as $a(x + d)(x + f)$.

Note that you don't have to use this method, since you can always use the quadratic formula, but it can be quicker in simple cases, for example, when d and f are integers. Even if d and f are integers, it can be hard to spot the factorization, so in an exam situation it is best not to spend too much time on this method. If d and f involve fractions, square roots or complex numbers, then it is usually almost impossible to spot the factorization, even for me. I also find it difficult if a is any number apart from one, so in these cases it is best to factor out the a, then perform the factorization and then multiply it back in to one of the factors at the end.

So let us look at factorizing expressions of the form $x^2 + bx + c$. If we want $x^2 + bx + c = (x + d)(x + f)$ then we must have $x^2 + bx + c = x^2 + (d + f)x + df$. Comparing the coefficients on either side of this equation, we see that we must have $d + f = b$ and $df = c$. That is we need to find numbers d and f whose sum is b and whose product is c.

I think the best way to proceed is to look at $df = c$ first and **ONLY** look for whole number possibilities for d and f. To see how it works let us have a look at $x^2 + x - 12$. In this case we need $-12 = df$. Now if we are looking for whole number values of d and f, then the possibilities are

$$(d, f) = (1, -12), (-1, 12), (2, -6), (-2, 6), (3, -4), (-3, 4).$$

There are the corresponding possibilities with d and f reversed, but we don't need to consider these since they give the same factorization. Now, we also need $d + f = 1$, and the only possibility from the above that works is $(d, f) = (-3, 4)$. Thus the required factorization is $x^2 + x - 12 = (x + (-3))(x + 4) = (x - 3)(x + 4)$.

Here are some more examples.

Example 3.5.7.

- $x^2 + 3x + 2 = (x + 1)(x + 2)$.
- $x^2 - x - 6 = (x + 2)(x - 3)$.
- $x^2 + 3x = x(x + 3)$.
- $x^2 - 6x + 8 = (x - 2)(x - 4)$.
- $x^2 - 8x + 16 = (x - 4)^2$.

Please don't worry too much about this method. It is something that will get better with practice but remember you can always use the quadratic formula if you want.